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IMPLEMENTATION OF DERIVATIONS AND INVARIANT SUBSPACES

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ABSTRACT

The paper studies operator implementations of derivations of algebras. Let π and ϱ be irreducible representations of an algebra $\mathcal{A}$ on Banach spaces X and Y. A linear map $\delta: \mathcal{A} \to B(Y,X)$ is a (π,ρ)-*derivation* if $\delta(ab) = \pi(a)\delta(b)+\delta(a)\rho(b)$. It is bimodule-closable if $\pi(a_n) \to 0, \rho(a_n) \to 0$ and $\delta(a_n) \to B$ imply $B = 0$. A closed operator F from Y into X implements δ if $F\rho(a) - \pi(a)F \subseteq \delta(a)$, for $a \in \mathcal{A}$. It is shown that if X, Y are reflexive and either the closure of the algebra $\{\pi(a) + \rho(a) : a \in \mathcal{A}\}$ or both algebras $\pi(\mathcal{A}), \rho(\mathcal{A})$ contain compact operators, then the set $\mathrm{Imp}(\delta)$ of all implementations is not empty for any bimodule-closable (π,ρ)-derivation δ, and either contains a *minimal* operator, or a

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maximal operator, or two families of operators $R_\lambda \subseteq G_\lambda$, $\lambda \in \mathbb{C}$, such that $R_\lambda \subseteq T \subseteq G_\lambda$ for each $T \in \mathrm{Imp}(\delta)$ and some λ.

These results are applied to the study of norm-closed operator algebras $\mathcal{B}$ on Banach spaces X with only one invariant subspace L. It is proved that, if $\mathcal{B}$ contains compact operators, X is reflexive and L has approximation property, then $\mathcal{B}$ contains all compact "corner" operators: $BX \subseteq L$ and $BL = 0$. If L has a closed complement, the same is true if the closure of the block-diagonal part of $\mathcal{B}$ contains compact operators. If X is non-reflexive, $\mathcal{B}$ may have no "corner" operators. If, however, $\mathcal{B}$ consists of compact operators then its weak closure contains all "corner" operators. A description is given of algebras of compact operators on Hilbert spaces with only one invariant subspace.

Introduction

Let X and Y be Banach spaces. We denote by $B(X)$ the algebra of all bounded operators on X and by $B(Y, X)$ the space of all bounded operators from Y into X. Let π and ρ be representations of an algebra $\mathcal{A}$ on X and Y, respectively. A (π, ρ)-**derivation** is a linear map δ from $\mathcal{A}$ into $B(Y, X)$ satisfying the rule:

$$\delta(ab) = \pi(a)\delta(b) + \delta(a)\rho(b).$$

Clearly, any (π, ρ)-derivation is a usual, spatial derivation from $\mathcal{A}$ into the $\mathcal{A}$-bimodule $B(Y, X)$. A (π, ρ)-derivation is called **bimodule-closable** if

$$\pi(a_n) \to 0, \rho(a_n) \to 0 \text{ and } \delta(a_n) \to B \in B(Y, X) \text{ imply that } B = 0.$$

Throughout the paper the convergence is in the norm topology unless another topology is indicated.

Each operator F in $B(Y, X)$ defines a bimodule-closable (π, ρ)-derivation δ_F of $\mathcal{A}$:

$$\delta_F(a) = \pi(a)F - F\rho(a) \quad \text{for all } a \in \mathcal{A}.$$

More generally, a densely defined operator F from Y to X **implements** a (π, ρ)-derivation δ of $\mathcal{A}$ if its domain $D(F)$ is ρ-invariant and if

$$(0.1) \qquad \delta(a)|_{D(F)} = (F\rho(a) - \pi(a)F)|_{D(F)} \quad \text{for each } a \in \mathcal{A}.$$

We denote by $\mathrm{Imp}(\delta)$ the set of all closed, densely-defined operators which implement δ. It is not difficult to see that any implemented derivation must be bimodule-closable; we are interested in the conditions under which the converse is true.

The question "which unbounded derivations of an algebra $\mathcal{A}$ are implemented by densely defined operators" is of a cohomological nature. Its "bounded" version — "which derivations of $\mathcal{A}$ are implemented by bounded operators" — is the problem of the description of the first cohomology group of $\mathcal{A}$ with coefficients in the bimodule $B(Y, X)$.

Bratteli and Robinson [BR] studied the case where $X = Y$ is a Hilbert space and δ is a closable $*$-derivation of a $*$-algebra $\mathcal{A}$ in $B(X)$. They proved that, if the closure of $\mathcal{A}$ contains the ideal of all compact operators, then $\text{Imp}(\delta) \neq \emptyset$. In Section 2 we shall extend their result to bimodule-closable (π, ρ)-derivations of arbitrary algebras provided that the Banach spaces X, Y are reflexive, π, ρ are irreducible, and either the closure of the algebra $\{\pi(a) + \rho(a) : a \in \mathcal{A}\}$ or both algebras $\pi(\mathcal{A})$ and $\rho(\mathcal{A})$ contain non-zero compact operators.

Earlier in Section 1 we shall consider various properties of $\mathcal{F}$-representations — irreducible, infinite-dimensional representations which contain non-zero finite-rank operators in their images. Their theory appears to be surprisingly close to the theory of finite-dimensional, irreducible representations. For example, as in the classic Schur lemma, the space of all intertwining operators for two $\mathcal{F}$-representations is either trivial or "one-dimensional".

Section 3 describes the structure of the set $\text{Imp}(\delta)$ when π, ρ are irreducible representations whose images contain non-zero compact operators. It is proved that $\text{Imp}(\delta)$ either contains a *minimal* operator such that all $T \in \text{Imp}(\delta)$ extend it, or it contains a *maximal* operator which extends every $T \in \text{Imp}(\delta)$, or it contains two families of operator $\{R_\lambda\}_{\lambda \in \mathbb{C}}, \{G_\lambda\}_{\lambda \in \mathbb{C}}$, $R_\lambda \subseteq G_\lambda$, such that any $T \in \text{Imp}(\delta)$ satisfies $R_\lambda \subseteq T \subseteq G_\lambda$ for some $\lambda \in \mathbb{C}$.

The most natural class of (π, ρ)-derivations consists of derivations of subalgebras $\mathcal{A}$ of $B(X)$ into $B(X)$, where π and ρ are the identity representations. Another class is constituted by "corner" derivations of $\mathcal{A}$: let L be a closed $\mathcal{A}$-invariant subspace of X, M be a closed complement of L in X, and Q be the projection on M along L. Then π: $A \to A|L$, ρ: $A \to QA|M$, $A \in \mathcal{A}$, are representations of $\mathcal{A}$ and δ: $A \to (1-Q)A|M$ is a (π, ρ)-derivation of $\mathcal{A}$. This allows us to apply the above results about derivations to the study of the structure of operator algebras with only one non-trivial invariant subspace.

Let $\mathcal{B}$ be a norm-closed algebra of operators on a Banach space X, and suppose that $\mathcal{B}$ contains a non-zero compact operator. If X is a reflexive space with the approximation property and $\mathcal{B}$ has a trivial invariant subspace lattice $(\{0\}, X)$, then (see [L] and also [RR]) $\mathcal{B}$ contains all compact operators on X. It is natural to ask what can be said about $\mathcal{B}$ if it only has one non-trivial invariant subspace L. In Section 4 we shall establish that, if X is reflexive and L has the approximation property, then $\mathcal{B}$ must contain all compact operators T such that $TX \subseteq L$ and $TL = 0$ — compact "corner" operators. If L has a closed complement, the same is true under a weaker condition: the closure of the "block-diagonal part" of $\mathcal{B}$ contains a non-zero compact operator. If, however, X is non-reflexive, then $\mathcal{B}$ may have no non-trivial operators vanishing on L.

It is also proved, without the assumption of reflexivity of X, that if $\mathcal{B}$ consists of compact operators then its weak closure contains all "corner" operators. We finish Section 4 by a description of algebras of compact operators on Hilbert spaces with only one non-trivial invariant subspace.

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1. Properties of $\mathcal{F}$-representations

We denote by $\mathcal{F}(X)$ the algebra of all finite-rank operators on a Banach space X, and by X^* the dual space of X. For $x \in X$ and $g \in X^*$, the rank-one operator $g \otimes x$ acts on X by

$$g \otimes x(z) = g(z)x \quad \text{for } z \in X.$$

For each operator A on X, we denote by $D(A)$ its domain and by A^* the conjugate operator on X^*. If A is closable, that is, $x_n \to 0$ and $Ax_n \to x$ imply that $x = 0$, then we denote by $\overline{A}$ its closure. If $x \in D(A)$ and $g \in D(A^*)$, then

$$A(g \otimes x) = g \otimes Ax \text{ and } (g \otimes x)A = A^*g \otimes x. \tag{1.1}$$

Hence

$$(g \otimes x)(h \otimes y) = g(y)(h \otimes x), \quad \text{so that } (g \otimes x)^2 = g(x)(g \otimes x). \tag{1.2}$$

If $g(x) \neq 0$ then $g \otimes tx$ is a rank-one projection for some $t \in \mathbb{C}$.

Let $\mathcal{U}$ be a subalgebra of $B(X)$. Then $\mathcal{U}$ is *transitive* if its lattice of *closed* invariant subspaces consists only of $\{0\}$ and X. For each manifold L in X, we denote by $\mathcal{U}L$ the linear span of $\{Ax : A \in \mathcal{U}, x \in L\}$. Set

$$\mathcal{U}_{\mathcal{F}} = \mathcal{U} \cap \mathcal{F}(X).$$

If $\mathcal{U}$ is transitive and $\mathcal{U}_{\mathcal{F}} \neq \{0\}$, then $\mathcal{U}_{\mathcal{F}}$ is also transitive on X and contains a rank-one projection ([B]). We need the following refinement of this result.

LEMMA 1.1: *Let $\mathcal{U}$ be a transitive subalgebra of $B(X)$. If $\mathcal{U}_{\mathcal{F}} \neq \{0\}$, then the ideal J generated by all rank-one projections in $\mathcal{U}$ coincides with $\mathcal{U}_{\mathcal{F}}$.*

Proof: We prove the lemma by induction on the rank $r(T)$ of operators T. If $r(T) = 0$, then $T = 0 \in J$. Assume that J contains all operators with rank smaller than k, and let $T \in \mathcal{U}_{\mathcal{F}}$ with $r(T) = k$.

Let $x = Ty \neq 0$. There is $S \in \mathcal{U}_{\mathcal{F}}$ with $Sx \neq 0$. Let R be a rank-one operator with $RSx = y$, and set $P = TRS$. Then $Px = x$. Since $\mathcal{U}_{\mathcal{F}}$ is transitive, it is weakly dense in $B(X)$ (see Theorem 8.23 in [RR]), so that $T\mathcal{U}_{\mathcal{F}}S$ is weakly dense in $TB(X)S$. Since $TB(X)S$ is finite-dimensional, $TB(X)S = T\mathcal{U}_{\mathcal{F}}S \subseteq \mathcal{U}_{\mathcal{F}}$. Hence $P \in \mathcal{U}_{\mathcal{F}}$. Since $r(P) = 1$ and $Px = x$, P is a rank-one projection, so that $P \in J$. We have $T = PT + (1-P)T$ and $PT \in J$. Since $r((1-P)T) < r(T)$, we have $(1-P)T \in J$. Hence $T \in J$. ∎

Definition 1.2: An irreducible representation π of an algebra $\mathcal{A}$ on X is called an $\mathcal{F}$-representation if $\pi(\mathcal{A}) \cap \mathcal{F}(X) \neq \{0\}$.

For a representation π of $\mathcal{A}$ on X, we set

$$I_\pi = \{a \in \mathcal{A} : \pi(a) \in \mathcal{F}(X)\}. \tag{1.3}$$

Then I_π is an ideal of $\mathcal{A}$ and $\mathrm{Ker}(\pi) \subseteq I_\pi$. If π is an $\mathcal{F}$-representation of $\mathcal{A}$, then the operator algebra $\mathcal{U} = \pi(\mathcal{A})$ is transitive, $\mathrm{Ker}(\pi) \subset I_\pi$ and

$$\mathcal{U}_{\mathcal{F}} = \pi(\mathcal{A}) \cap \mathcal{F}(X) = \pi(I_\pi) \neq \{0\}.$$

Consider the subspaces

$$E_\pi = \pi(I_\pi)X \quad \text{and} \quad E_\pi^* = \pi(I_\pi)^* X^*.$$

LEMMA 1.3: *Let π be an $\mathcal{F}$-representation of an algebra $\mathcal{A}$ on X. Then*

(i) *E_π is dense in X and contained in any non-zero π-invariant subspace of X,*

(ii) *$E_\pi^* \neq \{0\}$ is contained in any non-zero π^*-invariant subspace of X^*.*

Proof: The subspace E_π is non-zero and π-invariant. Hence it is dense in X. If L is a non-zero, π-invariant subspace of X, it is dense in X. Hence, for any $a \in \mathcal{A}$, $\pi(a)L$ is dense in $\pi(a)X$. If $a \in I_\pi$, then $\dim \pi(a)X < \infty$, so that $\pi(a)X = \pi(a)L \subseteq L$. Hence $E_\pi \subseteq L$. Part (i) is proved.

Set $\mathcal{R} = \{r \in \mathcal{A} : \pi(r)$ is a rank-one operator$\}$. It follows from Lemma 1.1 that $\pi(I_\pi)$ coincides with the linear manifold generated by all operators $\pi(r)$

with $r \in \mathcal{R}$. Let L be a non-zero π^*-invariant subspace of X^*. To prove (ii) it suffices to show that $\pi(r)^* X^* \subseteq L$ for each $r \in \mathcal{R}$.

Let $r \in \mathcal{R}$ and $\pi(r) = g \otimes x$, where $0 \neq x \in X$ and $0 \neq g \in X^*$. Then $\pi(r)^* = x \otimes g$ and $\pi(r)^* X^* = \mathbb{C}g$. For each $a \in \mathcal{A}$, $ar \in \mathcal{R}$ and $\pi(ar) = \pi(a)\pi(r) = g \otimes \pi(a)x$. Let $0 \neq h \in L$. Then $\pi(ar)^* h = (\pi(a)x \otimes g)h = h(\pi(a))g \in L$. Since π is irreducible, there exists $a \in \mathcal{A}$ such that $h(\pi(a)) \neq 0$. Hence $\pi(r)^* X^* = \mathbb{C}g \subseteq L$. ∎

It follows from Lemma 1.3 that

$$E_\pi = \pi(I_\pi)x = \pi(\mathcal{A})y \quad \text{and} \quad E_\pi^* = \pi(I_\pi)^* f = \pi(\mathcal{A})^* g, \tag{1.4}$$

for any $0 \neq x \in X$ and $0 \neq y \in E_\pi$, any $0 \neq f \in X^*$ and $0 \neq g \in E_\pi^*$.

LEMMA 1.4: *Let π be a representation of $\mathcal{A}$, and let J be an ideal of $\mathcal{A}$ not contained in* $\mathrm{Ker}(\pi)$.

(i) *If π is irreducible, then the representation $\sigma = \pi|J$ is irreducible.*

(ii) *If π is an $\mathcal{F}$-representation, then σ an $\mathcal{F}$-representation and $E_\sigma = E_\pi$.*

Proof: The representation σ irreducible, since, for each $x \in X$, we have

$$\overline{\pi(J)x} \supseteq \overline{\pi(\mathcal{A})\pi(J)\pi(\mathcal{A})x} = \overline{\pi(\mathcal{A})\pi(J)X} = X.$$

The representation $\pi|I_\pi$ is irreducible, whence $\overline{\pi(J)\pi(I_\pi)X} = \overline{\pi(J)X} = X$. Since $\pi(J)\pi(I_\pi) \subseteq \pi(J) \cap \mathcal{F}(X)$, we have $\pi(J) \cap \mathcal{F}(X) \neq \{0\}$. Hence σ is an $\mathcal{F}$-representation.

Since $I_\sigma = J \cap I_\pi$, we have $E_\sigma \subseteq E_\pi$. On the other hand E_σ is π-invariant and, by Lemma 1.3(i), $E_\pi \subseteq E_\sigma$. Thus $E_\pi = E_\sigma$. ∎

If π is an $\mathcal{F}$-representation of $\mathcal{A}$, then there is $p \in \mathcal{A}$ such that $\pi(p)$ is a rank-one projection. For later investigations it is important to know the conditions when, for two $\mathcal{F}$-representations π, ρ of $\mathcal{A}$, there exists an element p in $\mathcal{A}$ such that both $\pi(p)$ and $\rho(p)$ are rank-one projections.

We call $\mathcal{F}$-representations π, ρ **coherent** if

$$\rho(I_\pi) \neq \{0\} \quad \text{and} \quad \pi(I_\rho) \neq \{0\}. \tag{1.5}$$

THEOREM 1.5: *Let π and ρ be $\mathcal{F}$-representations of $\mathcal{A}$ on X and Y, respectively. There exists $p \in \mathcal{A}$ such that $\pi(p)$ and $\rho(p)$ are rank-one projections if and only if π and ρ are coherent.*

Proof: Let π and ρ be coherent $\mathcal{F}$-representations. Without loss of generality, we suppose that $\mathrm{Ker}(\pi) \cap \mathrm{Ker}(\rho) = \{0\}$. If π, ρ are not faithful, then, by Lemma 1.4,

$\pi|\operatorname{Ker}(\rho), \rho|\operatorname{Ker}(\pi)$ are $\mathcal{F}$-representations. Thus there are $a \in \operatorname{Ker}(\pi)$, $b \in \operatorname{Ker}(\rho)$ such that $\pi(b)$ and $r(a)$ are rank-one projections. It remains to set $p = a + b$.

Assume now that π is faithful. There is $a \in \mathcal{A}$ such that $\rho(a)$ is a rank-one projection. Clearly, $\pi(a) \neq 0$. There is also $b \in \mathcal{A}$ such that $\rho(b) \neq 0$ and $\pi(b)$ has rank one. Indeed, $\pi(\operatorname{Ker}(\rho))$ is an ideal of $\pi(\mathcal{A})$. If it contains all rank one projections in $\pi(\mathcal{A})$, then, by Lemma 1.1, it contains $\pi(\mathcal{A}) \cap \mathcal{F}(X) = \pi(I_\pi)$. Since π is faithful, $I_\pi \subseteq \operatorname{Ker}(\rho)$, which contradicts (1.5).

Clearly, $r(\pi(axb)) \leq 1$ and $r(\rho(axb)) \leq 1$ for each $x \in \mathcal{A}$. Since $\rho(A)$ is transitive, there is $x \in \mathcal{A}$ with $\rho(axb) \neq 0$. Since π is faithful, $\pi(axb) \neq 0$. Thus we have found an element $c \in \mathcal{A}$ such that $\pi(c)$ and $\rho(c)$ are rank-one operators, say

$$\pi(c) = g \otimes e \text{ and } \rho(c) = h \otimes f, \quad \text{where } e \in X, g \in X^*, f \in Y \text{ and } h \in Y^*.$$

Set $\mathcal{A}_1 = \{a \in \mathcal{A} : g(\pi(a)e) = 0\}$, $\mathcal{A}_2 = \{a \in \mathcal{A} : h(\rho(a)f) = 0\}$. Then $\mathcal{A}_i$ are proper subspaces of $\mathcal{A}$, so that $\mathcal{A} \neq \mathcal{A}_1 \cup \mathcal{A}_2$. Hence there is $b \in \mathcal{A}$ such that $g(\pi(b)e) \neq 0$ and $h(\rho(b)f) \neq 0$. Taking (1.1) and (1.2) into account, we have that $\pi(bc) = g \otimes \pi(b)e$ and $\rho(bc) = h \otimes \rho(b)f$ are non-nilpotent rank-one operators. Hence there is $0 \neq t \in \mathbb{C}$ such that the element $p = tbc$ satisfies $\pi(p)^2 = \pi(p)$. Since π is faithful, $p^2 = p$, whence $\rho(p)$ is also a rank-one projection.

The converse is obvious. ∎

Remark 1.6: The following conditions are sufficient for $\mathcal{F}$-representations π, ρ to be coherent:

(a) $\operatorname{Ker}(\pi) = \operatorname{Ker}(\rho)$;

(b) $\operatorname{Ker}(\pi)$ is not contained in $\operatorname{Ker}(\rho)$ and $\operatorname{Ker}(\rho)$ is not contained in $\operatorname{Ker}(\pi)$.

Indeed, if $\operatorname{Ker}(\pi) = \operatorname{Ker}(\rho)$ and $\pi(I_\rho) = 0$, then $\rho(I_\rho) = 0$, which is impossible for an $\mathcal{F}$-representation. Sufficiency of (b) was established in Theorem 1.5, but it is easy to prove it directly: if π, ρ are not coherent, say $\pi(I_\rho) = 0$, then $\operatorname{Ker}(\rho) \subseteq \operatorname{Ker}(\pi)$. ∎

LEMMA 1.7: *Let π be an $\mathcal{F}$-representation, and suppose that ρ is irreducible. If $\operatorname{Ker}(\rho) = \operatorname{Ker}(\pi)$, then ρ is also an $\mathcal{F}$-representation.*

Proof: Without loss of generality, we may assume that both π and ρ are faithful. Let $p \in \mathcal{A}$ be such that $\pi(p)$ is a rank-one projection. Then $\pi(p\mathcal{A}p)$ is one-dimensional. Since π, ρ are faithful, the same is true for $p\mathcal{A}p$ and $p^2 = p$, so $\rho(p)$ is a projection. Since $\rho(\mathcal{A})$ is transitive, $\rho(p)\mathcal{A}x = \rho(p\mathcal{A}p)x$ is dense in $\rho(p)X$

for each $x \in \rho(p)X$. Hence $\rho(p)$ has rank one, so that ρ is an $\mathcal{F}$-representation. ∎

LEMMA 1.8: *Let π and ρ be coherent $\mathcal{F}$-representations of $\mathcal{A}$ on X and Y, respectively, and let δ be a (π, ρ)-derivation of $\mathcal{A}$. Then any densely defined operator T which implements δ (see (0.1)) is closable.*

Proof: By Theorem 1.5, there is $p \in \mathcal{A}$ such that $\pi(p) = g \otimes e$ and $\rho(p) = h \otimes f$ are rank-one projections, where $e \in X$, $g \in X^*$, $f \in Y$ and $h \in Y^*$. Then $\pi(p)e = g(e)e = e$. Since $D(T)$ is ρ-invariant, $\rho(p)y = h(y)f$ belongs to $D(T)$ for $y \in D(T)$. Since $D(T)$ is dense in Y, $f \in D(T)$.

Let $y_n \to 0$ in Y and $Ty_n \to x$ in X. For each $a \in \mathcal{A}$, we have $\rho(a)y_n \to 0$. By (0.1),

$$g(\pi(a)x)e = \pi(p)\pi(a)x = \lim \pi(pa)Ty_n = \lim \delta(pa)y_n + \lim T\rho(pa)y_n$$
$$= \lim T\rho(p)\rho(a)y_n = \lim h(\rho(a)y_n)Tf = 0.$$

Hence $g(\pi(a)x) = 0$ for all $a \in \mathcal{A}$. Since π is irreducible, $x = 0$. ∎

2. Existence of implementations of bimodule-closable derivations

Let π and ρ be representations of an algebra $\mathcal{A}$ on Banach spaces X and Y and let $\mathcal{D} = \{\pi(a)\dot{+}\rho(a) : a \in \mathcal{A}\}$ be the corresponding operator algebra on $X\dot{+}Y$. In this section we prove the following generalization of the Bratteli–Robinson theorem (see [BR]).

THEOREM 2.0: *Let π and ρ be irreducible representations of $\mathcal{A}$ and let X and Y be reflexive Banach spaces. If the norm closure of the operator algebra $\mathcal{D}$ contains a non-zero, compact operator, then any bimodule-closable (π, ρ)-derivation of $\mathcal{A}$ is implemented by a closed, densely defined operator.*

We will prove Theorem 2.0 in a few steps. First we require some auxiliary results.

LEMMA 2.1: *Let δ be a (π, ρ)-derivation of $\mathcal{A}$.*

(i) *If a closable operator F implements δ, then $\overline{F} \in \mathrm{Imp}(\delta)$.*

(ii) *If $\mathrm{Imp}(\delta) \neq \emptyset$, then δ is bimodule-closable.*

Proof: Let $x_n \in D(F)$, $x_n \to x \in D(\overline{F})$ and $Fx_n \to \overline{F}x$. For $a \in \mathcal{A}$,

$$\rho(a)x_n \to \rho(a)x \quad \text{and} \quad F\rho(a)x_n = \delta(a)x_n + \pi(a)Fx_n \to \delta(a)x + \pi(a)\overline{F}x.$$

Hence $\rho(a)x \in D(\overline{F})$ and $\delta(a)x = \overline{F}\rho(a)x - \pi(a)\overline{F}x$. Thus $\overline{F} \in \mathrm{Imp}(\delta)$ and (i) is proved.

Let $R \in \mathrm{Imp}(\delta)$, $\pi(a_n) \to 0$, $\rho(a_n) \to 0$ and $\delta(a_n) \to B$. For $y \in D(R)$, we have

$$By = \lim \delta(a_n)y = \lim(R\rho(a_n)y - \pi(a_n)Ry) = \lim R\rho(a_n)y.$$

Since R is closed, $By = 0$. Thus $B = 0$, so that δ is bimodule-closable. ∎

LEMMA 2.2: *Let δ be a (π, ρ)-derivation of $\mathcal{A}$, let J be an ideal of $\mathcal{A}$, and suppose that* $\mathrm{Imp}(\delta|J) \neq \emptyset$.

(i) *If ρ is irreducible and J is not contained in* $\mathrm{Ker}(\rho)$, *then* $\mathrm{Imp}(\delta) \neq \emptyset$.

(ii) *If π, ρ are irreducible and J is not contained in* $\mathrm{Ker}(\pi) \cap \mathrm{Ker}(\rho)$, *then* $\mathrm{Imp}(\delta) \neq \emptyset$.

Proof: If $T \in \mathrm{Imp}(\delta|J)$, then $\rho(J)D(T) \subseteq D(T)$. By Lemma 1.4, $\rho|J$ is irreducible, so that $\rho(J)D(T)$ is dense in Y. By (0.1), for each $a \in \mathcal{A}$, $b \in J$, we have

$$\begin{aligned}\delta(a)\rho(b)x &= \delta(ab)x - \pi(a)\delta(b)x = \pi(ab)Tx - T\rho(ab)x - \pi(a)[\pi(b)Tx - T\rho(b)x] \\ &= (T\rho(a) - \pi(a)T)\rho(b)x\end{aligned}$$

whenever $x \in D(T)$. Hence $T' = T|\rho(J)D(T)$ is a densely defined closable operator which implements δ. By Lemma 2.1(i), $\overline{T'} \in \mathrm{Imp}(\delta)$.

Taking (i) into account, we may suppose that $J \subseteq \mathrm{Ker}(\rho)$. Then $\delta(b)y = \pi(b)Ty$ for each $y \in D(T)$ and $b \in J$. The subspace

$$G = \{x \dot{+} y \in X \dot{+} Y : \delta(b)y = \pi(b)x \text{ for } b \in J\}$$

is closed in $X \dot{+} Y$ and contains the graph $\{Ty \dot{+} y : y \in D(T)\}$ of T. If $x \dot{+} 0 \in G$, then $\pi(b)x = 0$ for $b \in J$. Since $\mathrm{Ker}(\pi)$ does not contain J, it follows from Lemma 1.4 that $\pi(J)$ is transitive. Hence $x = 0$, so that G is a graph of a closed operator S: $G = \{y \dot{+} Sy : y \in D(S)\}$ and $\delta(b)y = \pi(b)Sy$ for $y \in D(S)$ and $b \in J$.

The subspace $D(S)$ is ρ-invariant. Indeed, for $a \in \mathcal{A}$, $b \in J$ and $y \in D(S)$,

$$\delta(b)(\rho(a)y) = \delta(ba)y - \pi(b)\delta(a)y = \pi(b)(\pi(a)y - \delta(a)y).$$

Therefore

$$\pi(b)(S\rho(a)y) = \delta(b)(\rho(a)y) = (\delta(ba)y - \pi(b)\delta(a)y) = \pi(b)(\pi(a)Sy - \delta(a)y).$$

Since $\pi(J)$ is transitive, $\delta(a)y = \pi(a)Sy - S\rho(a)y$. Thus $S \in \mathrm{Imp}(\delta)$. ∎

Clearly, if δ is a bimodule-closable (π, ρ)-derivation, then

$$\mathrm{Ker}(\pi) \cap \mathrm{Ker}(\rho) \subseteq \mathrm{Ker}(\delta). \tag{2.1}$$

The following result represents the first step in the proof of Theorem 2.0, and also shows that, for coherent $\mathcal{F}$-representations π, ρ, each (π, ρ)-derivation satisfying (2.1) is bimodule-closable.

THEOREM 2.3: *Let π, ρ be coherent $\mathcal{F}$-representations, and let δ be a (π, ρ)-derivation such that* $\mathrm{Ker}(\pi) \cap \mathrm{Ker}(\rho) \subseteq \mathrm{Ker}(\delta)$. *Then* $\mathrm{Imp}(\delta) \neq \emptyset$.

Proof: By replacing $\mathcal{A}$ by $\mathcal{A}/(\mathrm{Ker}(\pi) \cap \mathrm{Ker}(\rho))$, we may suppose that $\mathrm{Ker}(\pi) \cap \mathrm{Ker}(\rho) = \{0\}$. By Theorem 1.5, there exists $p \in \mathcal{A}$ such that $\pi(p) = g \otimes e$ and $\rho(p) = h \otimes f$ are rank-one projections: $g(e) = f(h) = 1$. Since $p^2 - p$ belongs to $\mathrm{Ker}(\pi) \cap \mathrm{Ker}(\rho)$, p is a projection.

Set $C = p\mathcal{A}p$. The representations $\pi(C)$ and $\rho(C)$ are one-dimensional. Hence $\dim(C) \leq 2$, since $\mathrm{Ker}(\pi) \cap \mathrm{Ker}(\rho) = 0$. If $\dim(C) = 1$, then $C = \mathbb{C}p$. As in the proof of Theorem 8 in [BR], setting $T = \delta(p)$, $\delta_T(a) = T\rho(a) - \pi(a)T$ and $\Delta = \delta - \delta_T$, we obtain that Δ is a (π, ρ)-derivation and $\Delta(p) = 0$. Therefore $\Delta(C) = 0$.

Now suppose that $\dim(C) = 2$. Then $C = \mathbb{C}p + \mathbb{C}q$, where $\pi(q) = 0$ and $\rho(p - q) = 0$. Setting $T = \delta(p)$ and $\Delta' = \delta - \delta_T$ as above, we have $\Delta'(p) = 0$. Now set $S = \Delta'(q)$ and $\Delta = \Delta' - \delta_S$. Since $pq = qp = q$, we have

$$\Delta'(q) = \Delta'(pq) = \pi(p)\Delta'(q) \quad \text{and} \quad \Delta'(q) = \Delta'(qp) = \Delta'(q)\rho(p).$$

Therefore, taking into account the fact that $\rho(q) = \rho(p)$, we obtain

$$\begin{aligned} \Delta(p) &= \Delta'(p) - (\Delta'(q)\rho(p) - \pi(p)\Delta'(q)) = 0, \\ \Delta(q) &= \Delta'(q) - (\Delta'(q)\rho(q) - \pi(q)\Delta'(q)) = \Delta'(q) - \Delta'(q)\rho(p) = 0. \end{aligned}$$

Thus $\Delta(C) = 0$.

The condition that $\Delta(pap) = 0$ for $a \in \mathcal{A}$ gives $\pi(p)\Delta(a)\rho(p) = 0$. Making use of (1.1) and (1.2), we have $g(\Delta(a)f) = 0$. Applying this in the case where $a = cb$, we obtain

$$g(\pi(c)\Delta(b)f) + g(\Delta(c)\rho(b)f) = 0 \quad \text{for } b, c \in \mathcal{A}.$$

If $\rho(b)f = 0$, for some b in $\mathcal{A}$, then $g(\pi(c)\Delta(b)f) = 0$, for all $c \in \mathcal{A}$, and hence $\Delta(b)f = 0$, since $\pi(\mathcal{A})$ is transitive. This allows us to define a linear operator

F: $F(\rho(b)f) = \Delta(b)f$ on the subspace $L = \rho(\mathcal{A})f$, which is dense in Y. The operator F implements Δ:

$$\Delta(a)(\rho(b)f) = \Delta(ab)f - \pi(a)\Delta(b)f = (F\rho(a) - \pi(a)F)(\rho(b)f).$$

By Lemma 1.8, F is closable, so $\overline{F} \in \mathrm{Imp}(\Delta)$, which implies that $\mathrm{Imp}(\delta) \neq \emptyset$. ∎

Let π, ρ be $\mathcal{F}$-representations, δ be a (π, ρ)-derivation, and let $T \in \mathrm{Imp}(\delta)$. Then $D(T)$ is ρ-invariant and $D(T^*)$ is π^*-invariant. By Lemma 1.3, $E_\rho \subseteq D(T)$ and $E_\pi^* \subseteq D(T^*)$. Clearly, $\overline{T|E_\rho} \in \mathrm{Imp}(\delta)$ and, in the case where both X and Y are reflexive,

$$\overline{T|E_\rho} \subseteq T \subseteq (T^*|E_\pi^*)^*.$$

LEMMA 2.4: *If X, Y are reflexive, then $(T^*|E_\pi^*)^* \in \mathrm{Imp}(\delta)$.*

Proof: Let $A \in B(X)$, $B \in B(Y)$ and $C \in B(Y, X)$ be such that

$$BD(T) \subseteq D(T) \quad \text{and} \quad AT + TB \subseteq C.$$

A standard argument shows that

$$A^*D(T^*) \subseteq D(T^*) \quad \text{and} \quad T^*A^* + B^*T^* \subseteq C^*. \tag{2.2}$$

Applying this to the inclusion $T\rho(a) - \pi(a)T \subseteq \delta(a)$, we obtain

$$\pi(a)^*D(T^*) \subseteq D(T^*) \quad \text{and} \quad \rho(a)^*T^* - T^*\pi(a)^* \subseteq \delta(a)^* \quad \text{for each } a \in \mathcal{A}.$$

Taking into account the fact that E_π^* is π^*-invariant and contained in $D(T^*)$, denote $T^*|E_\pi^*$ by S. Then $\rho(a)^*S - S\pi(a)^* \subseteq \delta(a)^*$ and, since X, Y are reflexive, $S^*\rho(a) - \pi(a)S^* \subseteq \delta(a)$. This means that $S^* \in \mathrm{Imp}(\delta)$. ∎

THEOREM 2.5: *Let π and ρ be irreducible representations of $\mathcal{A}$, and let δ be a bimodule-closable (π, ρ)-derivation.*

(i) *If $\mathrm{Ker}(\pi) = \mathrm{Ker}(\rho)$ and π or ρ is an $\mathcal{F}$-representation, then $\mathrm{Imp}(\delta) \neq \emptyset$.*
(ii) *If $\mathrm{Ker}(\pi)$ is not contained in $\mathrm{Ker}(\rho)$ and ρ is an $\mathcal{F}$-representation, then $\mathrm{Imp}(\delta) \neq \emptyset$.*
(iii) *Suppose that X and Y are reflexive. If $\mathrm{Ker}(\rho)$ is not contained in $\mathrm{Ker}(\pi)$ and π is an $\mathcal{F}$-representation, then $\mathrm{Imp}(\delta) \neq \emptyset$.*

Proof: By Remark 1.6 and Lemma 1.7, both π and ρ in (i) are coherent $\mathcal{F}$-representations. Hence (i) follows from Theorem 2.3.

Suppose that $J = \mathrm{Ker}(\pi)$ is not contained in $\mathrm{Ker}(\rho)$. Denote by ρ', δ' the restrictions of ρ, δ to J. By Lemma 2.2, in order to prove (ii) we need to show that $\mathrm{Imp}(\delta') \neq \emptyset$. It follows from Lemma 1.4 that ρ' is an $\mathcal{F}$-representation. Since δ is bimodule-closable,

$$\mathrm{Ker}(\rho') = \mathrm{Ker}(\pi) \cap \mathrm{Ker}(\rho) \subseteq \mathrm{Ker}(\delta').$$

Replacing J by $J/\mathrm{Ker}(\rho')$, we may suppose that ρ' is faithful.

Let $p \in J$ be such that $\rho'(p) = h \otimes f$ is a rank-one projection. If $\rho'(b)f = 0$ for some $b \in J$, then $\rho'(bp) = 0$. Hence $bp = 0$, so that

$$\delta'(b)f = \delta'(b)\rho'(p)f = \delta'(bp) = 0.$$

As in Theorem 2.3, this allows us to define a linear operator F: $F(\rho'(b)f) = \delta'(b)f$ on the subspace $L = \rho'(J)f$ which is dense in Y such that F implements δ'.

To show that F is closable, assume that $\rho'(b_n)f \to 0$ and $\delta'(b_n)f \to x$. Then $\rho'(b_n p) \to 0$ and $\delta'(b_n p) = \delta'(b_n)\rho'(p) \to h \otimes x$. Since δ' is bimodule-closable, $h \otimes x = 0$, so that $x = 0$. Part (ii) is proved.

Set $J = \mathrm{Ker}(\rho)$, and let δ', π' be the restrictions of δ, π to J. By Lemma 1.4, π' is an $\mathcal{F}$-representation. Since δ is bimodule-closable,

$$\mathrm{Ker}(\pi') = \mathrm{Ker}(\pi) \cap \mathrm{Ker}(\rho) \subseteq \mathrm{Ker}(\delta').$$

Replacing J by $J/\mathrm{Ker}(\pi')$, we assume that π' is faithful. We have

$$\delta'(bc) = \pi'(b)\delta'(c) \quad \text{for } b, c \in J.$$

Let $p \in J$ be such that $\pi(p) = g \otimes e$ is a rank-one projection. As in (ii), the operator S: $\pi'(b)^*g \to \delta'(b)^*g$ from $D = \pi'(J)^*g \subseteq X^*$ into Y^* is well defined and closable. For each $a \in \mathcal{A}$, we have

$$\begin{aligned}\delta(a)^*(\pi'(b)^*g) &= [\delta(ba) - \delta(b)\rho(a)]^*g \\ &= S\pi'(ba)^*g - \rho(a)^*S\pi'(b)^*g = [S\pi(a)^* - \rho(a)^*S](\pi'(b)^*g).\end{aligned}$$

Hence $S\pi(a)^* - \rho(a)^*S \subseteq \delta(a)^*$. Set $T = -S^*$. Taking into account the fact that X and Y are reflexive, we obtain from (2.2) that $T \in \mathrm{Imp}(\delta)$. ∎

COROLLARY 2.6: *Let π and ρ be representations of $\mathcal{A}$ on reflexive Banach spaces X and Y, respectively.*

(i) *If* $\mathrm{Ker}(\pi) \cap \mathrm{Ker}(\rho) \neq I_\pi \cap I_\rho$ *(see (1.3)), then* $\mathrm{Imp}(\delta) \neq \emptyset$ *for each bimodule-closable (π, ρ)-derivation δ.*

(ii) *If π and ρ are $\mathcal{F}$-representations, then* $\mathrm{Imp}(\delta) \neq \emptyset$ *for each bimodule-closable (π, ρ)-derivation δ.*

Proof: Let $a \in I_\pi \cap I_\rho$ and $a \notin \mathrm{Ker}(\pi) \cap \mathrm{Ker}(\rho)$. If both operators $\pi(a)$ and $\rho(a)$ are non-zero, then π and ρ are coherent $\mathcal{F}$-representations and (i) follows from Theorem 2.3. If $\pi(a) \neq 0$ and $\rho(a) = 0$, then π is an $\mathcal{F}$-representation and $\mathrm{Ker}(\rho)$ is not contained in $\mathrm{Ker}(\pi)$, so that (i) follows from Theorem 2.5(iii). In the remaining case, (i) follows from Theorem 2.5(ii).

Similarly, part (ii) follows from Theorem 2.5. ∎

Remark 2.7: The proof of Theorem 2.5(iii) was based on the reduction to the case $\rho = 0$. The example below shows that, if the spaces X, Y are not reflexive, then, for some $\mathcal{F}$-representations π, $(\pi, 0)$-derivations need not be implemented.

Let $Y = X$, $\mathcal{A} = \mathcal{F}(X)$, and $\pi(A) = A$ for $A \in \mathcal{A}$. Let T be a bounded operator on the second dual space X^{**} such that TX is not contained in X. Set $\delta(A) = A^{**}T|X$ for $A \in \mathcal{A}$. Since A^{**} maps X^{**} into X, $\delta(A) \in B(X)$. Clearly, δ is a bimodule-closable $(\pi, 0)$-derivation. Since $\mathcal{A}$ has no invariant linear subspaces, a closed operator S implementing δ would be everywhere defined and, hence, bounded. It follows that $S = T$, which is impossible.

The proof of the following result is standard; we include it for the reader's convenience.

Proposition 2.8: *Let $\mathcal{A}$ be a closed, unital subalgebra of $B(X)$, let φ be a bounded isomorphism from $\mathcal{A}$ into $B(X)$, and let $\mathrm{Sp}(A) = \mathrm{Sp}(\varphi(A))$ for $A \in \mathcal{A}$. If P is a projection in the norm-closure of $\varphi(\mathcal{A})$, then, for any $e > 0$, there is a projection Q_ε in $\varphi(\mathcal{A})$ such that $\|P - Q_\varepsilon\| < e$.*

Proof: Let U and V be disjoint closed disks centered at 0 and 1, respectively, and let L be the boundary of V. Then $\mathrm{Sp}(P) \subset U \cup V$. Since the spectrum function $B \to \mathrm{Sp}(B)$ is upper semicontinuous (see Theorem 3.4.2 in [A]), there exists $\delta > 0$ such that, for each $B \in B(X)$, $\|B - P\| < \delta$ implies that $\mathrm{Sp}(B) \subset U \cup V$.

Let $R(B, \lambda) = (B - \lambda 1)^{-1}$ and $C = \max_{\lambda \in L} \|R(P, \lambda)\|$. If $\|P - B\| < C^{-1}$, then

$$B - \lambda 1 = [1 - (P - B)R(P, \lambda)](P - \lambda 1) \quad \text{for each } \lambda \in L,$$

so that

$$\|R(B, \lambda)\| = \left\| R(P, \lambda) \sum_{n=0}^{\infty} [(P - B)R(P, \lambda)]^n \right\| \leq \frac{C}{1 - C\|P - B\|}.$$

Therefore

$$\|R(P,\lambda)-R(B,\lambda)\|=\|R(P,\lambda)(B-P)R(B,\lambda)\|\leq\frac{C^2\|P-B\|}{1-C\|P-B\|}.$$

For each $B\in B(X)$, consider the Riesz projection

$$Q(B)=-\frac{1}{2\pi i}\oint_L R(B,\lambda)d\lambda$$

(see I.2.3 in [GK]). We have $Q(P)=P$ and, by the above,

$$\|P-Q(B)\|=\|Q(P)-Q(B)\|\leq\frac{1}{2\pi}\oint_L\|R(P,\lambda)-R(B,\lambda)\|d\lambda\to 0,$$

if $\|P-B\|\to 0$.

Let $B=\varphi(\mathcal{A})$ for $A\in\mathcal{A}$. Then $\mathrm{Sp}(A)=\mathrm{Sp}(B)\subset U\cup V$ and its boundary $\partial\,\mathrm{Sp}(A)\subset U\cup V$. Let $\mathrm{Sp}_{\mathcal{A}}(A)$ be the spectrum of A in $\mathcal{A}$. Since $\mathcal{A}$ is a closed subalgebra of $B(X)$, we have $\partial\,\mathrm{Sp}_{\mathcal{A}}(A)\subseteq\partial\,\mathrm{Sp}(A)$ (see Theorem 3.2.13(ii) in [A]). Taking this into account, we obtain $\mathrm{Sp}_{\mathcal{A}}(A)\subset U\cup V$. Hence $R(A,\lambda)\in\mathcal{A}$, for each $\lambda\in L$, so that $R(B,\lambda)=\varphi(R(A,\lambda))$.

Since $\mathcal{A}$ is closed,

$$Q(A)=-\frac{1}{2\pi i}\oint_L R(A,\lambda)d\lambda\in\mathcal{A}.$$

Since $Q(A)$ is the limit of the Riemann sums and φ is bounded,

$$Q(B)=Q(\varphi(\mathcal{A}))=-\frac{1}{2\pi i}\oint_L\varphi(R(A,\lambda))d\lambda=\varphi(Q(A)).\tag{2.3}$$ ∎

Definition 2.9: A (π,ρ)-derivation δ of $\mathcal{A}$ is called bimodule-closed if

(i) $\mathrm{Ker}(\pi)\cap\mathrm{Ker}(\rho)\subseteq\mathrm{Ker}(\delta)$;

(ii) $\pi(a_n)\to A$, $\rho(a_n)\to B$ and $\delta(a_n)\to C$ imply that there is $a\in\mathcal{A}$ such that $\pi(a)=A$, $\rho(a)=B$, $\delta(a)=C$.

If δ is bimodule-closed, it is, clearly, bimodule-closable.

THEOREM 2.10: *Let π and ρ be irreducible representations of an algebra $\mathcal{A}$ with identity on X and Y, and let δ be a bimodule-closed (π,ρ)-derivation of $\mathcal{A}$. If the norm-closure of the operator algebra $\mathcal{D}=\{\pi(a)\dot{+}\rho(a):a\in\mathcal{A}\}$ in $B(X\dot{+}Y)$ contains a non-zero compact operator, then $\mathrm{Ker}(\pi)\cap\mathrm{Ker}(\rho)\neq I_\pi\cap I_\rho$ (see (1.3)), so that at least one of the representations π and ρ is an $\mathcal{F}$-representation.*

Proof: Since δ is bimodule-closed and $1\in\mathcal{A}$, the operator algebra

$$\mathcal{B}=\left\{\hat{a}=\begin{pmatrix}\pi(a)&\delta(a)\\0&\rho(a)\end{pmatrix}:a\in\mathcal{A}\right\}$$

on $Z = X\dot{+}Y$ is closed in $B(Z)$ and $1_Z \in \mathcal{B}$. The isomorphism φ: $\hat{a} \to \begin{pmatrix} \pi(a) & 0 \\ 0 & \rho(a) \end{pmatrix}$ from $\mathcal{B}$ onto $\mathcal{D}$ is bounded and $\mathrm{Sp}(\hat{a}) = \mathrm{Sp}(\varphi(\hat{a}))$.

Let

$$B = \begin{pmatrix} K & 0 \\ 0 & T \end{pmatrix}$$

be a compact operator in $\bar{\mathcal{D}}$ with $K \neq 0$. For each $a \in \mathcal{A}$,

$$B(a) = B\varphi(\hat{a}) = \begin{pmatrix} K\pi(a) & 0 \\ 0 & T\rho(a) \end{pmatrix} \in \bar{\mathcal{D}}.$$

Since $\pi(\mathcal{A})$ is transitive on X, it follows from Lemma 8.22 in [RR] that there is $a \in \mathcal{A}$ such that $1 \in \mathrm{Sp}(K\pi(a))$. Then $B(a)$ is compact and $1 \in \mathrm{Sp}(B(a))$. Let $P \neq 0$ be the finite-rank projection on the spectral subspace of $B(a)$ corresponding to the eigenvalue 1. Since $\bar{\mathcal{D}}$ is closed in $B(Z)$, P belongs to $\bar{\mathcal{D}}$.

By Proposition 2.8, there is $a \in \mathcal{A}$ such that

$$\varphi(\hat{a}) = \begin{pmatrix} \pi(a) & 0 \\ 0 & \rho(a) \end{pmatrix}$$

is a projection and $\|P - \varphi(\hat{a})\| < \frac{1}{2}$. Hence $0 \neq \varphi(\hat{a})$ is a finite-rank projection, so that $\pi(a)$ and $\rho(a)$ are finite-rank projections, and at least one of them is non-zero. Thus $a \in \mathrm{Ker}(\pi) \cap \mathrm{Ker}(\rho)$ and $a \in I_\pi \cap I_\rho$. ∎

Let δ be a (π, ρ)-derivation of $\mathcal{A}$, and set $Z = X\dot{+}Y$. Denote by $\tilde{\mathcal{A}}$ the closed operator subalgebra of $B(Z)$ generated by 1_Z and by all the operators $\begin{pmatrix} \pi(a) & \delta(a) \\ 0 & \rho(a) \end{pmatrix}$, where $a \in \mathcal{A}$. Let Q be the projection on Y along X. Then $\tilde{\pi}(A) := A|X$ and $\tilde{\rho}(A) := QA|Y$ are representations of $\tilde{\mathcal{A}}$ on X and Y, respectively, and $\tilde{\delta}(A) := (1_Z - Q)A|Y$ is a $(\tilde{\pi}, \tilde{\rho})$-derivation of $\tilde{\mathcal{A}}$. In a standard way, one proves the following result.

Lemma 2.11: *If π and ρ are irreducible and δ is bimodule-closable, then the derivation $\tilde{\delta}$ is bimodule-closed and* $\mathrm{Imp}(\tilde{\delta}) = \mathrm{Imp}(\delta)$. ∎

Finally, we shall conclude the proof of Theorem 2.0.

Proof of Theorem 2.0: The closure of the algebra $\{\pi(a)\dot{+}\rho(a) : a \in \mathcal{A}\}$ coincides with the closure of the algebra $\{\tilde{\pi}(A)\dot{+}\tilde{\rho}(A) : A \in \tilde{\mathcal{A}}\}$, and therefore contains a non-zero compact operator. Since $\tilde{\delta}$ is bimodule-closed, it follows from Corollary 2.6(i) and Theorem 2.10 that $\mathrm{Imp}(\tilde{\delta}) \neq \emptyset$. Applying now Lemma 2.11, we complete the proof. ∎

We denote by $\mathcal{K}(X)$ the ideal of all compact operators on X.

Definition 2.12: An irreducible representation is called a $\mathcal{K}$-**representation** if its image contains a non-zero compact operator.

COROLLARY 2.13: *Let π and ρ be $\mathcal{K}$-representations of $\mathcal{A}$ on X and Y.*

(i) *If $\mathcal{A}$ has identity and δ is a bimodule-closed (π,ρ)-derivation of $\mathcal{A}$, then* $\operatorname{Ker}(\pi)\cap\operatorname{Ker}(\rho)\neq I_\pi\cap I_\rho$ *(see (1.3)), so that at least one of the representations π and ρ is an $\mathcal{F}$-representation.*

(ii) *If X and Y are reflexive, then each bimodule-closable (π,ρ)-derivation of $\mathcal{A}$ is implemented by a closed operator.*

Proof: By Theorems 2.0 and 2.10, we need only show that there exists $c\in\mathcal{A}$ such that $\pi(c)\dot{+}\rho(c)$ is a non-zero compact operator. Let $\pi(a)$ and $\rho(b)$ be non-zero compact operators. If $\rho(a)=0$ and $\pi(b)=0$, then set $c=a+b$. If $\rho(a)\neq 0$ (the case $\pi(b)\neq 0$ is similar), then there exists $d\in\mathcal{A}$ such that $\rho(a)\rho(d)\rho(b)\neq 0$. In this case set $c=adb$. ∎

PROBLEM 2.14: *Does the conclusion of Theorem 2.0 hold if we weaken the condition that the closure of the algebra $\{\pi(a)\dot{+}\rho(a) : a\in\mathcal{A}\}$ contains a non-zero compact operator, and only assume that $\overline{\pi(\mathcal{A})}\cap\mathcal{K}(X)\neq\{0\}$ and $\overline{\rho(\mathcal{A})}\cap\mathcal{K}(Y)\neq\{0\}$?*

The next corollary extends the result of Proposition 3.4.9 in [S] (see also Theorem 3 in [BR]) to derivations of Banach algebras.

COROLLARY 2.15: *Let δ be a bimodule-closed (π,π)-derivation of an algebra $\mathcal{A}$ with identity and P be a projection in $\overline{\pi(\mathcal{A})}$. For any $\varepsilon>0$, there is $a_\varepsilon\in\mathcal{A}$ such that $\pi(a_\varepsilon)$ is a projection and $\|P-\pi(a_\varepsilon)\|\leq\varepsilon$.*

Proof: Without loss of generality, we may suppose that $\operatorname{Ker}(\pi)=\{0\}$. Since δ is bimodule-closed,

$$\mathcal{B}=\left\{\hat{a}=\begin{pmatrix}\pi(a) & \rho(a)\\ 0 & \pi(a)\end{pmatrix} : a\in\mathcal{A}\right\}$$

is a closed subalgebra of $B(X\dot{+}X)$ and $1\in\mathcal{B}$. The map $\varphi\colon\hat{a}\to\begin{pmatrix}\pi(a) & 0\\ 0 & \pi(a)\end{pmatrix}$ is a bounded isomorphism from $\mathcal{B}$ into $B(X\dot{+}X)$ and $\operatorname{Sp}(\hat{a})=\operatorname{Sp}(\varphi(\hat{a}))$.

The projection

$$\tilde{P}=\begin{pmatrix}P & 0\\ 0 & P\end{pmatrix}$$

belongs to $\overline{\varphi(\mathcal{B})}$. By Proposition 2.8, for each $\varepsilon>0$, there exists $a_\varepsilon\in\mathcal{A}$ such that $\varphi(\hat{a}_\varepsilon)$ is a projection and $\|\tilde{P}-\varphi(\hat{a}_\varepsilon)\|<\varepsilon$. Hence $\pi(a_\varepsilon)$ is a projection and $\|P-\pi(a_\varepsilon)\|<\varepsilon$. ∎

3. Structure of Imp(δ)

It is natural to begin the study of Imp(δ) with the case when $\delta = 0$. This case is the simplest one but, on the other hand, fundamental because, for any $T, S \in \mathrm{Imp}(\delta)$ with $D(T) \cap D(S) \neq \{0\}$, their difference implements $\delta = 0$ (in general, however, $T - S$ is not defined).

A linear operator T from Y into X **intertwines** representations π and ρ of $\mathcal{A}$ on X and Y respectively, if its domain $D(T)$ is ρ-invariant and

$$\pi(a)Ty = T\rho(a)y \quad \text{for } y \in D(T).$$

If π and ρ are irreducible and $T \neq 0$, then

$$\mathrm{Ker}(T) = 0, \quad D(T) \text{ is dense in } Y \quad \text{and} \quad TD(T) \text{ is dense in } X. \tag{3.1}$$

The set of all *closed* intertwining operators is denoted by $\mathrm{Int}(\pi, \rho)$. Thus $\mathrm{Int}(\pi, \rho) = \mathrm{Imp}(0)$.

We define the maps $\gamma: \pi(\mathcal{A}) \to \rho(\mathcal{A})$ and $\gamma': \rho(\mathcal{A}) \to \pi(\mathcal{A})$ by

$$\begin{aligned} \gamma(\pi(a)) &= \rho(a), \quad \text{if } \mathrm{Ker}(\pi) \subseteq \mathrm{Ker}(\rho); \\ \gamma'(\rho(a)) &= \pi(a), \quad \text{if } \mathrm{Ker}(\rho) \subseteq \mathrm{Ker}(\pi). \end{aligned} \tag{3.2}$$

For finite-dimensional irreducible representations, the classic Schur's lemma states that $\mathrm{Int}(\pi, \rho)$ is trivial, whenever $\mathrm{Ker}(\rho) \neq \mathrm{Ker}(\pi)$, and is a one-dimensional space otherwise. For $\mathcal{F}$-representations the situation is similar.

LEMMA 3.1:

(i) *Let π and ρ be irreducible. If $\mathrm{Ker}(\pi) \neq \mathrm{Ker}(\rho)$, then $\mathrm{Int}(\pi, \rho) = \{0\}$. Moreover, any operator intertwining ρ and π is zero.*

(ii) *Let π and ρ be $\mathcal{F}$-representations. If $\mathrm{Ker}(\pi) = \mathrm{Ker}(\rho)$, then*
   1. *there exists $0 \neq T_- \in \mathrm{Int}(\pi, \rho)$ such that any $T \in \mathrm{Int}(\pi, \rho)$ is an extension of λT_- for some $\lambda \in \mathbb{C}$;*
   2. *the maps γ and γ' are closable.*

Proof: If $0 \neq T$ intertwines π and ρ, then $\pi(a)TD(T) = \{0\}$ for $a \in \mathrm{Ker}(\rho)$, and $T\rho(b)D(T) = \{0\}$ for $b \in \mathrm{Ker}(\pi)$. Taking (3.1) into account, we have $\mathrm{Ker}(\pi) = \mathrm{Ker}(\rho)$. This proves (i).

Suppose that $\mathrm{Ker}(\pi) = \mathrm{Ker}(\rho)$. Then (see Remark 1.6) π and ρ are coherent, so that, by Theorem 1.5, there exists $p \in \mathcal{A}$ such that

$$\pi(p) = g \otimes e, \quad \rho(p) = h \otimes f \quad \text{with } g(e) = h(f) = 1.$$

If, for some $a \in \mathcal{A}$, $\rho(a)f = 0$, then $\rho(ap) = 0$. Hence $\pi(ap) = 0$, and so $\pi(a)e = 0$. This allows us to define a linear operator S on $E_\rho := \rho(\mathcal{A})f$ by setting $S\rho(a)f = \pi(a)e$ for $a \in \mathcal{A}$. Obviously S intertwines π and ρ. By Lemma 1.8, S is closable; we denote its closure by T_-.

Let $0 \neq R \in \mathrm{Int}(\pi, \rho)$. Then $f \in E_\rho \subseteq D(R)$. We have to prove that the restriction of R to E is proportional to S. By (1.1),

$$h \otimes \pi(a)Rf = h \otimes R\rho(a)f = R\rho(a)\rho(p) = \pi(a)\pi(p)R = R^*g \otimes \pi(a)e$$

for $a \in \mathcal{A}$. Hence $\pi(a)Rf = \lambda\pi(a)e$ for some $0 \neq \lambda \in \mathbb{C}$. Therefore $Rf = \lambda e$. From this it follows that $R|E_\rho = \lambda S$ because

$$R\rho(a)f = \pi(a)Rf = \lambda\pi(a)e = \lambda S\rho(a)e \quad \text{for } a \in \mathcal{A}.$$

Thus part (ii) (1) is proved. Part (2) follows from (1) and (3.1). ∎

Our next result shows in particular (when $\delta = 0$) that, for reflexive X, Y, there is also $\hat{T} \in \mathrm{Int}(\pi, \rho)$ such that any $T \in \mathrm{Int}(\pi, \rho)$ is proportional to a restriction of $\hat{T}$ to $D(T)$.

THEOREM 3.2: *Let π and ρ be $\mathcal{F}$-representations of $\mathcal{A}$ on reflexive Banach spaces X and Y, and let δ be a bimodule-closable (π, ρ)-derivation.*

(i) *If $\mathrm{Ker}(\rho) \neq \mathrm{Ker}(\pi)$, then there are operators $T_{\min}$ and $T_{\max}$ in $\mathrm{Imp}(\delta)$ such that $T_{\min} \subseteq T \subseteq T_{\max}$ for any $T \in \mathrm{Imp}(\delta)$.*

(ii) *If $\mathrm{Ker}(\rho) = \mathrm{Ker}(\pi)$, then there are closable operators S, F from E_ρ into X such that*

(1) *$0 \neq \overline{F} \in \mathrm{Int}(\pi, \rho)$ and $\overline{S} \in \mathrm{Imp}(\delta)$;*

(2) *for each $\lambda \in \mathbb{C}$, the operators $S + \lambda F$ are closable and the operators $R_\lambda := \overline{S + \lambda F}$ and $G_\lambda := ((S + \lambda F)^*|E_\pi^*)^*$ belong to $\mathrm{Imp}(\delta)$;*

(3) *for each $T \in \mathrm{Imp}(\delta)$, there exists $\lambda \in \mathbb{C}$ such that $R_\lambda \subseteq T \subseteq G_\lambda$.*

Proof: By Corollary 2.6, there exists $K \in \mathrm{Imp}(\delta)$. By Lemma 1.3, $E_\rho \subseteq D(T)$ for each $T \in \mathrm{Imp}(\delta)$. The operator $S := K|E_\rho$ implements δ, so, by Lemma 2.1, $\overline{S} \in \mathrm{Imp}(\delta)$. Clearly, the operator $R(T) = T|E_\rho - S$ intertwines π and ρ.

If $\mathrm{Ker}(\rho) \neq \mathrm{Ker}(\pi)$, it follows from Lemma 3.1 that $R(T) = 0$, so T extends S. We have $T^* \subseteq S^*$. Since $D(T^*)$ is π^*-invariant, it follows from Lemma 1.3 that $E_\pi^* \subseteq D(T^*)$. Hence $(T^*|E_\pi^*)^* = (S^*|E_\pi^*)^*$. By Lemma 2.4, $(T^*|E_\pi^*)^* \in \mathrm{Imp}(\delta)$. Since $T \subseteq (T^*|E_\pi^*)^*$, we have $S \subseteq T \subseteq (S^*|E_\pi^*)^*$, and so, to finish the proof of (i), it only remains to set $T_{\min} = \bar{K}'$ and $T_{\max} = ((K')^*|E_\pi^*)^*$.

If $\mathrm{Ker}(\rho) = \mathrm{Ker}(\pi)$, then, by Lemma 3.1, there exists $0 \neq T_- \in \mathrm{Int}(\pi, \rho)$. Set $F = T_-|E_\rho$. Then (1) is satisfied. The operators $S + \lambda F$ implement δ for

$\lambda \in \mathbb{C}$. Since, by Remark 1.6, π and ρ are coherent representations, it follows from Lemmas 1.8 and 2.1 that $S + \lambda F$ are closable operators and $R_\lambda \in \mathrm{Imp}(\delta)$. By Lemma 2.4, G_λ also belong to $\mathrm{Imp}(\delta)$.

We obtain from the above discussion and Lemma 3.1 that, for any $T \in \mathrm{Imp}(\delta)$, there exists $t \in \mathbb{C}$ such that $R(T) = T|E_\rho - S = tF$. Thus $T|E_\rho = R_\lambda|E_\rho$. Hence

$$R_\lambda \subseteq \overline{T|E_\rho} \subseteq T \subseteq (T^*|E_\pi^*)^* = ((T|E_\rho)^*|E_\pi^*)^* = (R_\lambda^*|E_\pi^*)^* = G_\lambda,$$

as required. ∎

The examples below illustrate both possibilities.

Example 3.3: Let R and S be closed densely defined operators from Y into X such that $R \subseteq S$. Consider the algebra

$$\mathcal{A} = \left\{ A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \in B(X \dot{+} Y) : A_2 D(S) \subseteq D(R), \right.$$
$$\left. A_{12}|_{D(S)} = (SA_2 - A_1 S)|_{D(S)} \right\},$$

and set $\pi(A) = A_1$, $\rho(A) = A_2$, and $\delta(A) = A_{12}$. Then π and ρ are $\mathcal{F}$-representations of $\mathcal{A}$, and δ is a bimodule-closed (π, ρ)-derivation. The algebra $\mathcal{A}$ is reflexive, and the lattice of invariant subspaces of $\mathcal{A}$ consists of $\{0\}$, X, $X \dot{+} Y$ and all L such that $G(R) \subseteq L \subseteq G(S)$, where $G(R)$ and $G(S)$ are the graphs of R and S. Hence $R = T_{\min}$ is the smallest implementation of δ and $S = T_{\max}$ is its largest implementation. ∎

Example 3.4 [K]*:* Let R and T be densely defined, closed operators from Y into X such that:

(1) $D(R) \cap D(T)$ is dense in Y and $D(R^*) \cap D(T^*)$ is dense in X^*;

(2) $\mathrm{Ker}(T) = \{0\}$ and TY is dense in X.

Then, for each $\lambda \in \mathbb{C}$, the operators $R + \lambda T$ and $R^* + \bar{\lambda} T^*$ are closable. Set $R_\lambda = \overline{R + \lambda T}$ and $S_\lambda = (R^* + \bar{\lambda} T^*)^*$, and consider the operator algebra

$$\mathcal{A} = \left\{ A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \in B(X \dot{+} Y) : 1) A_2 D(R) \subseteq D(R), A_2 D(T) \subseteq D(T); \right.$$
$$\left. 2) A_1 T|_{D(T)} = TA_2|_{D(T)};\ 3) A_{12}|_{D(R)} = (RA_2 - A_1 R)|_{D(R)} \right\}.$$

Set $\pi(A) = A_1$, $\rho(A) = A_2$ and $\delta(A) = A_{12}$. Then π and ρ are $\mathcal{F}$-representations of $\mathcal{A}$ and δ is a bimodule-closed (π, ρ)-derivation. It was proved in Theorem 3.5 in [K] that: (1) all operators R_λ and S_λ belong to $\mathrm{Imp}(\delta)$; and (2) an operator

$G \in \mathrm{Imp}(\delta)$ if and only if $D(G)$ is ρ-invariant and $R_\lambda \subseteq G \subseteq S_\lambda$ for some $\lambda \in \mathbb{C}$. ∎

We will prove now that, if π and ρ are $\mathcal{K}$-representations (see Definition 2.12), then the structure of $\mathrm{Imp}(\delta)$ in many respects remains the same as for $\mathcal{F}$-representations.

THEOREM 3.5: *Let π and ρ be $\mathcal{K}$-representations of $\mathcal{A}$ on reflexive Banach spaces X and Y, and let δ be a bimodule-closable (π, ρ)-derivation. Suppose that*

(3.3) $\mathrm{Ker}(\pi) = \mathrm{Ker}(\rho)$ *and the maps γ, γ' (see (3.2)) are closable.*

Then there are $S \in \mathrm{Imp}(\delta)$, $F \in \mathrm{Int}(\pi, \rho)$, and $D \subseteq X^*$ such that

(i) $R_\lambda = \overline{S + \lambda F} \in \mathrm{Imp}(\delta)$ *and* $G_\lambda = ((S+\lambda F)^*|D)^* \in \mathrm{Imp}(\delta)$ *for each* $\lambda \in \mathbb{C}$;

(ii) *for any $T \in \mathrm{Imp}(\delta)$, there exists $\lambda \in \mathbb{C}$ such that $R_\lambda \subseteq T \subseteq G_\lambda$.*

Otherwise there are two possibilities:

(1) *there is $T_{\min} \in \mathrm{Imp}(\delta)$ such that $T_{\min} \subseteq T$ for any $T \in \mathrm{Imp}(\delta)$;*

(2) *there is $T_{\max} \in \mathrm{Imp}(\delta)$ such that $T \subseteq T_{\max}$ for any $T \in \mathrm{Imp}(\delta)$.*

Proof: It follows from Lemma 2.11 that there exist a unital Banach algebra $\tilde{\mathcal{A}}$ with representations $\tilde{\pi}$ and $\tilde{\rho}$ on X and Y and a bimodule-closed $(\tilde{\pi}, \tilde{\rho})$-derivation $\tilde{\delta}$ of $\tilde{\mathcal{A}}$ such that $\pi(\mathcal{A}) \subseteq \tilde{\pi}(\tilde{\mathcal{A}})$, $\rho(\mathcal{A}) \subseteq \tilde{\rho}(\tilde{\mathcal{A}})$, and $\mathrm{Imp}(\delta) = \mathrm{Imp}(\tilde{\delta})$. We also have $\mathrm{Int}(\pi, \rho) = \mathrm{Int}(\tilde{\pi}, \tilde{\rho})$. Moreover, (3.3) holds if and only if $\mathrm{Ker}(\tilde{\pi}) = \mathrm{Ker}(\tilde{\rho})$ and the maps $\tilde{\gamma}(\tilde{\pi}(\tilde{a})) = \tilde{\rho}(\tilde{a})$ and $\tilde{\gamma}'(\tilde{\rho}(\tilde{a})) = \tilde{\pi}(\tilde{a})$ are closable for all $\tilde{a} \in \tilde{\mathcal{A}}$. Thus, without loss of generality, we may suppose that δ is bimodule-closed.

By Corollary 2.13, $\mathrm{Imp}(\delta) \neq \emptyset$ and $\mathrm{Ker}(\pi) \cap \mathrm{Ker}(\rho) \neq I_\pi \cap I_\rho$, so that at least one of π and ρ is an $\mathcal{F}$-representation.

If (3.3) holds, then, by Lemma 1.7, both π and ρ are $\mathcal{F}$-representations and the proof follows from Theorem 3.2(ii).

Suppose now that (3.3) does not hold. If both π and ρ are $\mathcal{F}$-representations, it follows from Theorem 3.2(i) that $\mathrm{Imp}(\delta)$ satisfies both (1) and (2).

Suppose that ρ is an $\mathcal{F}$-representation and π is not. Then $\mathrm{Ker}(\pi) = I_\pi$. Since

$$\mathrm{Ker}(\pi) \cap \mathrm{Ker}(\rho) \neq I_\pi \cap I_\rho = \mathrm{Ker}(\pi) \cap I_\rho,$$

there is $a \in J$ such that $0 \neq \rho(a)$ is a finite-rank operator. Set $J = \mathrm{Ker}(\pi)$. By Lemma 1.4, $\rho' := \rho|J$ is an $\mathcal{F}$-representation and $E_\rho = E_{\rho'}$. It follows from (1.4) that, for each $0 \neq y \in E_\rho$,

$$E_\rho = E_{\rho'} = \rho'(J)y = \rho(J)y.$$

Let $0 \neq K \in \mathrm{Imp}(\delta)$. Then $D(K)$ is ρ-invariant, so that, by Lemma 1.3(i), E_ρ is dense in Y and $E_\rho \subseteq D(K)$. Set $R = K|E_\rho$. Then $\delta(a)|E_\rho = R\rho(a)|E_\rho$ for each $a \in J$. Therefore, for each $0 \neq y \in E_\rho$, we have

$$\delta(b)(\rho(a)y) = \delta(ba)y - \pi(b)\delta(a)y = (R\rho(b) - \pi(b)R)(\rho(a)y),$$

for $a \in J$, $b \in \mathcal{A}$. Since $D(R) = E_\rho = \rho(J)y$ is dense in Y, it follows that R implements δ. Hence, by Lemma 2.1(i), $\overline{R} \in \mathrm{Imp}(\delta)$.

For any $T \in \mathrm{Imp}(\delta)$, $D(T)$ is ρ-invariant, so that $E_\rho \subseteq D(T)$ and

$$\delta(a)|E_\rho = R\rho(a)|E_\rho = T\rho(a)|E_\rho \quad \text{for each } a \in J.$$

Hence $(R - T)\rho(J)E_\rho = \{0\}$, so that $T|E_\rho = R$. Setting $T_{\min} = \overline{R}$, we have $T_{\min} \subseteq T$ for each $T \in \mathrm{Imp}(\delta)$.

Similarly, one can show that, if π is an $\mathcal{F}$-representation and ρ is not, then there is $T_{\max} \in \mathrm{Imp}(\delta)$ such that $T \subseteq T_{\max}$ for each $T \in \mathrm{Imp}(\delta)$. ∎

4. Implementing operators and invariant subspaces

In this section we investigate the structure of norm-closed operator algebras $\mathcal{B}$ on Banach spaces X with only one non-trivial invariant subspace $L \subseteq X$. We impose some compactness conditions on $\mathcal{B}$ without which even the class of transitive operator algebras on X seems to be indescribable.

To clarify the situation, let us consider the case where $\dim X < \infty$. In this case, for an appropriate basis in X, the algebra $\mathcal{B}$ either consists of all block-matrices $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ or of all block-matrices $\begin{pmatrix} A & C \\ 0 & A \end{pmatrix}$ (this is a simple special case of Theorem 4.9 below). In both cases $\mathcal{B}$ contains the space $\mathfrak{C}_L$ of all matrices $\begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}$, and decomposes into the direct sum of $\mathfrak{C}_L$ and the block-diagonal part. It should be noted that $\mathfrak{C}_L$ has a simple, basis-independent description

$$\mathfrak{C}_L = \{A \in B(X) : AL = \{0\}, AX \subseteq L\},$$

and it is isomorphic to $B(X/L, L)$. In the general case, we aim to prove that $\mathcal{B}$ has a non-zero intersection with $\mathfrak{C}_L$, which implies that $\mathcal{B} \cap \mathfrak{C}_L$ is transitive or even weakly dense in $\mathfrak{C}_L$.

We consider now an arbitrary operator algebra $\mathcal{B}$ on X. Let L be a non-trivial invariant subspace of $\mathcal{B}$. Denote by φ_L the standard homomorphism from $\mathcal{B}$ into $B(X/L) : \varphi_L(A)(x + L) = Ax + L$, and set

$$\mathcal{B}|L = \{A|L : A \in \mathcal{B}\}, \quad \varphi_L(\mathcal{B}) = \{\varphi_L(A) : A \in \mathcal{B}\}.$$

In what follows the terms "weakly closed" and "weakly dense" mean closed or dense in the weak operator topology (WOT) on $B(X)$.

LEMMA 4.1: *Let $\mathcal{B}|L$ and $\varphi_L(\mathcal{B})$ be transitive algebras, and suppose that at least one of them contains a compact operator. If $\mathcal{B}\cap\mathfrak{C}_L \neq \{0\}$, then $\mathcal{B}\cap\mathfrak{C}_L$ is weakly dense in $\mathfrak{C}_L$.*

Proof: Set $\hat{X} = X/L$. For $T \in \mathfrak{C}_L$, define an operator $\tilde{T}$ in $B(\hat{X}, L)$: $\tilde{T}(x+L) = Tx$, for $x \in X$. Then $T \to \tilde{T}$ is an isometric, WOT-bicontinuous map from $\mathfrak{C}_L$ onto $B(\hat{X}, L)$. The image E of $\mathcal{B}\cap\mathfrak{C}_L$ in $B(\hat{X}, L)$ is a left $\mathcal{B}|L$- and a right $\varphi_L(\mathcal{B})$-module. Hence $\overline{E}^{wot}$ is a left $\overline{\mathcal{B}|L}^{wot}$- and a right $\overline{\varphi_L(\mathcal{B})}^{wot}$-module. Since the algebras $\mathcal{B}|L$ and $\varphi_L(\mathcal{B})$ are transitive, and at least one of them contains a compact operator, it follows from Theorem 8.23 in [RR] that either $\overline{\mathcal{B}|L}^{wot} = B(L)$, or $\overline{\varphi_L(\mathcal{B})}^{wot} = B(\hat{X})$. Hence $\overline{E}^{wot}$ contains a rank-one operator, say $f\otimes x$, where $x \in L$, $f \in \hat{X}^*$ and, therefore, all rank-one operators $(A|L)(f\otimes x)\varphi_L(B) = \varphi_L(B)^* f \otimes Ax$, for $A, B \in \mathcal{B}$, belong to $\overline{E}^{wot}$. Since the algebras $\mathcal{B}|L$ and $\varphi_L(\mathcal{B})$ are transitive, $\overline{E}^{wot}$ contains all rank-one operators. Thus $\overline{E}^{wot} = B(\hat{X}, L)$, so that $\mathcal{B}\cap\mathfrak{C}_L$ is weakly dense in $\mathfrak{C}_L$. ∎

Assume now that the invariant subspace L has a closed complement M in X. Let Q be the projection on M along L and consider the representations π: $A \to A|L$ and ρ: $A \to QA|M$ of $\mathcal{B}$ on L and M. Then δ: $A \to (1-Q)A|M$ is a (π,ρ)-derivation of $\mathcal{B}$.

We denote by $\mathcal{L}(\delta)$ the set of all invariant subspaces of $\mathcal{B}$ apart from $\{0\}$, L and X. Let F be an operator from M into L with domain $D(F) \subseteq M$. Its graph $G(F) = \{(Fy, y) : y \in D(F)\}$ is a subspace in X; it is closed if and only if F is closed.

LEMMA 4.2: *If π and ρ are irreducible representations, then $F \leftrightarrow G(F)$ is a bijection of* $\operatorname{Imp}(\delta)$ *onto $\mathcal{L}(\delta)$.*

Proof: By (0.1), $G(F) \in \mathcal{L}(\delta)$ if $F \in \operatorname{Imp}(\delta)$. Let $K \in \mathcal{L}(\delta)$. Since π is irreducible, either $L \subset K$, or $L\cap K = \{0\}$. Since ρ is irreducible, in the first case $K = X$ and in the second case there is a closed, densely defined operator F from M into L such that $K = G(F)$. Since $G(F)$ is invariant for all operators from $\mathcal{B}$, F implements δ. ∎

Note that under the isomorphism between M and X/L the algebra $\mathcal{B}_M = \rho(\mathcal{B}) = \{QA|M : A \in \mathcal{B}\}$ corresponds to $\varphi_L(\mathcal{B})$.

THEOREM 4.3: *Let $\mathcal{B}$ be a norm-closed algebra of operators on a reflexive Banach space X. Suppose that $\mathcal{B}$ has only one non-trivial invariant subspace L and that L has a closed complement M in X. If*
either

(i) *the closure of the "block-diagonal part" $\{A(1-Q)+QAQ : A \in B\}$ of $\mathcal{B}$ contains a non-zero compact operator,*

or

(ii) *the algebras $\mathcal{B}|L$ and $\mathcal{B}_M$ contain non-zero compact operators,*

then $\mathcal{B} \cap \mathfrak{C}_L$ is weakly dense in $\mathfrak{C}_L$.

Proof: Since L is the only non-trivial invariant subspace of $\mathcal{B}$, π and ρ are irreducible. Assume that $\mathcal{B} \cap \mathfrak{C}_L = \{0\}$. Then δ is bimodule-closable. Since L and M are reflexive, it follows from Theorem 2.0 and Corollary 2.13 that $\mathrm{Imp}(\delta) \neq \emptyset$. By Lemma 4.2, $\mathcal{L}(\delta) \neq \emptyset$, so that $\mathcal{B}$ has another non-trivial invariant subspace apart from L. This contradiction shows that $\mathcal{B} \cap \mathfrak{C}_L \neq \{0\}$.

By Theorem 2.10 and Corollary 2.13, at least one of the representations π and ρ is an $\mathcal{F}$-representation. Hence the weak density of $\mathcal{B} \cap \mathfrak{C}_L$ in $\mathfrak{C}_L$ follows from Lemma 4.1. ∎

Recall that by $\mathcal{K}(X)$ we denote the ideal of all compact operators on X. For any subspace L in X, the space

$$L^{\perp} = \{h \in X^* : h(y) = 0 \text{ for all } y \in L\}$$

in X^* is closed in $\sigma(X^*, X)$-topology. To study the case where L has no closed complement in X and X is non-reflexive, we consider the following pivotal result.

PROPOSITION 4.4: *Let $\mathcal{B}$ be a norm-closed subalgebra of $B(X)$ with only one non-trivial invariant subspace L, and suppose that $\mathcal{B} \cap \mathcal{K}(X) \neq \{0\}$.*

(i) *If $\mathcal{B} \cap \mathcal{K}(X)$ does not lie in $\mathfrak{C}_L$, then there is a $\mathcal{B}^*$-invariant, closed subspace $\mathfrak{L} \neq \{0\}$ in X^* such that $\mathcal{B}$ contains all operators $f \otimes x$, where $f \in \mathfrak{L}$, $x \in L$.*

(ii) *If $\varphi_L(\mathcal{B} \cap \mathcal{K}(X)) \neq 0$, then, in addition, $\mathfrak{L} \cap L^{\perp} \neq \{0\}$.*

Proof: Since L is the only non-trivial invariant subspace of $\mathcal{B}$, the algebras $\mathcal{B}|L$ and $\varphi_L(\mathcal{B})$ are transitive. Let us prove first that $\mathcal{B}$ contains a compact operator T such that $1 \in \mathrm{Sp}(T)$. If $K \in \mathcal{B} \cap \mathcal{K}(X)$ and $K|L \neq 0$, then, since the algebra $\mathcal{B}|L$ is transitive on L, it follows from [L] (see also [RR]) that there exists $A \in \mathcal{B}$ with $1 \in \mathrm{Sp}(KA|L)$. The operator $T := KA$ is compact and $1 \in \mathrm{Sp}(T)$. Suppose that $\varphi_L(K) \neq 0$. Since $\varphi_L(K)$ is compact and $\varphi_L(\mathcal{B})$ is a transitive algebra on X/L, we have similarly from [L] that there is $A \in \mathcal{B}$ with

$$1 \in \mathrm{Sp}(\varphi_L(K)\varphi_L(A)) = \mathrm{Sp}(\varphi_L(KA)) \subseteq \mathrm{Sp}(KA).$$

Thus again it suffices to set $T = KA$.

Let $P = Q(T)$ (see (2.3)) be the Riesz projection on the spectral subspace Z of T corresponding to $\{1\}$. Then $\dim Z < \infty$. Since $\mathcal{B}$ is norm-closed, $P \in \mathcal{B}$. Set $Z_L = Z \cap L$. Since $PL \subseteq L$, we have $PL = Z_L$. The algebra $P\mathcal{B}P|Z$ has no invariant subspaces apart from $\{0\}$, Z_L, and Z. Indeed, since L is the only non-trivial invariant closed subspace of $\mathcal{B}$,

(1) if $0 \neq z \in Z_L$, then $\mathcal{B}z$ is dense in L, so that $P\mathcal{B}Pz = Z_L$;

(2) if $0 \neq z \in Z$ and $z \notin Z_L$, then $\mathcal{B}z$ is dense in X, so that $P\mathcal{B}Pz = Z$;

and the claim follows.

If $Z_L = \{0\}$ or $Z_L = Z$, the algebra $P\mathcal{B}P|Z$ is transitive and, by the Burnside Theorem, $P\mathcal{B}P|Z = B(Z)$. Hence it contains a rank-one operator $g \otimes z$. If $\{0\} \neq Z_L \neq Z$, the same conclusion follows from Theorem 4.3 applied to the algebra $P\mathcal{B}P|Z$.

Since the set $\{x \in X : g \otimes x \in B\}$ is a closed $\mathcal{B}$-invariant subspace of X, it contains L. Similarly, the set $\mathfrak{L} = \{f \in X^* : f \otimes x \in \mathcal{B} \text{ for all } x \in L\}$ is a non-zero, closed subspace of X^*. This proves (i).

Assume now that $\varphi_L(\mathcal{B} \cap \mathcal{K}(X)) \neq 0$. As above, there is a compact operator T in $\mathcal{B}$ with $1 \in \mathrm{Sp}(\varphi_L(T)) \subseteq \mathrm{Sp}(T)$. Since φ_L is bounded, it follows from (2.3) that $\varphi_L(Q(T)) = Q(\varphi_L(T)) \neq 0$ is the Riesz projection onto the spectral subspace of $\varphi_L(T)$ corresponding to $\{1\}$. Hence Z does not lie in L, so $Z_L \neq Z$.

Suppose that $Z_L = \{0\}$ and $0 \neq g \otimes z \in P\mathcal{B}P|Z$. Then $z \in Z$. For $x \in L$, $(g \otimes z)x = g(x)z$. Since $z \notin L$ and L is invariant for $g \otimes z$, we have $g \in L^\perp$. Thus $\mathfrak{L} \cap L^\perp \neq \{0\}$.

Let $\{0\} \neq Z_L \neq Z$. Applying Theorem 4.3 to $P\mathcal{B}P|Z$, we obtain that there are $z \in Z_L$ and $g \in X^*$ such that $g \otimes z \in P\mathcal{B}P|Z$ and $g(x) = 0$ for $x \in Z_L$. Since $g \otimes z = (g \otimes z)P = P^*g \otimes z$, we have $g = P^*g$. Since $PL = Z_L$, we have, for $y \in L$,

$$g(y) = P^*g(y) = g(Py) = 0.$$

Thus $g \in L^\perp$, so that $\mathfrak{L} \cap L^\perp \neq \{0\}$. ∎

For each subspace $\mathfrak{M}$ in X^*, we denote by $\mathfrak{M} \otimes L$ the linear span of all rank-one operators $f \otimes x$, $f \in \mathfrak{M}$, $x \in L$. It is evident that $L^\perp \otimes L \subseteq \mathfrak{C}_L$.

THEOREM 4.5: *Let $\mathcal{B}$ be a norm-closed subalgebra of $B(X)$ which contains a non-zero compact operator, and suppose that $\mathcal{B}$ has only one non-trivial invariant subspace L.*

(i) *If the algebra $\mathcal{B}$ is either* (1) *weakly closed, or* (2) $\varphi_L(\mathcal{B} \cap \mathcal{K}(X)) \neq \{0\}$, *or* (3) X *is reflexive, then*

$$\mathcal{B} \cap \mathfrak{C}_L \neq \{0\}.$$

(ii) *If either* (1) $\varphi_L(\mathcal{B} \cap \mathcal{K}(X)) \neq \{0\}$ *(in particular, if $\mathcal{B} \subseteq \mathcal{K}(X)$), or* (2) $(\mathcal{B} \cap \mathcal{K}(X))|L \neq \{0\}$ *and X is reflexive,*

then $\mathcal{B} \cap \mathfrak{C}_L$ is weakly dense in $\mathfrak{C}_L$.

(iii) *If $\mathcal{B}$ is weakly closed and $\mathcal{B} \cap \mathcal{K}(X)$ does not lie in $\mathfrak{C}_L$, then $\mathfrak{C}_L \subset \mathcal{B}$.*

Proof: Part (i) follows from (ii) and (iii). Since L is the only non-trivial invariant subspace of $\mathcal{B}$, the algebras $\mathcal{B}|L$ and $\varphi_L(\mathcal{B})$ are transitive. Suppose that $\mathcal{B} \cap \mathcal{K}(X)$ is not contained in $\mathfrak{C}_L$. Then at least one of the algebras $\mathcal{B}|L$ and $\varphi_L(\mathcal{B})$ contains a non-zero compact operator, and it follows from Proposition 4.4 that there is a $\mathcal{B}^*$-invariant, norm closed subspace $\mathfrak{L} \neq \{0\}$ in X^* such that $\mathfrak{L} \otimes L \subseteq \mathcal{B}$.

Let $\varphi_L(\mathcal{B} \cap \mathcal{K}(X)) \neq \{0\}$. By Proposition 4.4(ii), $\mathfrak{L} \cap L^\perp \neq \{0\}$. Therefore $\{0\} \neq \mathcal{B} \cap (L^\perp \otimes L) \subseteq \mathcal{B} \cap \mathfrak{C}_L$ and part (ii) (1) follows from Lemma 4.1.

Let $\mathcal{B} \cap \mathcal{K}(X)$ contain an operator K such that $K|L \neq 0$. If X is reflexive, the only $\mathcal{B}^*$-invariant subspaces of X^* are $\{0\}$, $L^\perp$, and X^*. Since $\mathfrak{L} \neq \{0\}$, it is either L or X^*. Thus $L^\perp \otimes L \subseteq \mathfrak{L} \otimes L \subseteq \mathcal{B} \cap \mathfrak{C}_L$ and (ii) (2) follows from Lemma 4.1.

Let $\mathcal{B}$ be weakly closed and $\bar{\mathfrak{L}}^w$ be the closure of $\mathfrak{L}$ in the $\sigma(X^*, X)$-topology. Then $\bar{\mathfrak{L}}^w \otimes L \subseteq \mathcal{B}$. The space $\bar{\mathfrak{L}}^w$ is $\mathcal{B}^*$-invariant and, by the bipolar theorem, there is a norm closed subspace M in X such that $\bar{\mathfrak{L}}^w = M^\perp$. The space M is $\mathcal{B}$-invariant. Since $\mathfrak{L} \neq \{0\}$, M is either $\{0\}$ or L. In both cases $L^\perp \subseteq \bar{\mathfrak{L}}^w$, so $L^\perp \otimes L \subseteq \mathcal{B}$. Applying Lemma 4.1, we complete the proof. ∎

The reflexivity of X in Theorem 4.5(i) (3) and (ii) (2) is essential as the following example shows.

Example 4.6: Let H be a Hilbert space, $X = B(H)$ and $L = \mathcal{K}(H)$ be the ideal of all compact operators on H. Then X is the second dual of L. Let $B(L)$ be the algebra of all bounded operators on L. Set $\mathcal{B} = \{A^{**} : A \in B(L)\}$.

Then L is $\mathcal{B}$-invariant, $A^{**}|L = A$ for any $A \in B(L)$, and $\|A^{**}\| = \|A\|$. Hence $\mathcal{B}$ is a norm-closed subalgebra of $B(X)$ and

$$\mathcal{B} \cap \mathfrak{C}_L = \{0\}.$$

If $A \in B(L)$ is a rank-one operator, then A^{**} is also a rank-one operator.

Let us show that L is the only non-trivial invariant subspace of $\mathcal{B}$. For $B \in B(H)$, the operators λ_B, μ_B of left and right multiplication by B belong to $B(X)$, preserve L and $\lambda_B = (\lambda_B|L)^{**}$, $\mu_B = (\mu_B|L)^{**}$. Hence $\lambda_B, \mu_B \in \mathcal{B}$ and, by Calkin's Theorem, L is the only non-trivial invariant subspace of $\mathcal{B}$.

Remark 4.7: The above construction can be considered for any non-reflexive Banach space L: the algebra $\mathcal{B} = B(L)^{**}$ on L^{**} always contains non-zero compact operators and $\mathcal{B} \cap \mathfrak{C}_L = \{0\}$. However, for some L, $\mathcal{B}$ has other non-trivial invariant subspaces apart from L. An example of such a space is $L = c_0 \dot{+} l^1$.

We consider now the case when an operator algebra $\mathcal{B}$ consists of compact operators only.

COROLLARY 4.8: *Let $\mathcal{B}$ be an algebra of compact operators on X with only one non-trivial invariant space L. Then:*

(i) *$\bar{\mathcal{B}}^{wot}$ contains $\mathfrak{C}_L$;*

(ii) *if, in addition, X/L is reflexive and L has the approximation property, then $\mathfrak{C}_L \cap \mathcal{K}(X) \subseteq \mathcal{B}$.*

Proof: Part (i) follows from Theorem 4.5(ii) (1).

By Proposition 4.4(ii), $\mathcal{B}$ contains $\mathfrak{L}_1 \otimes L$, where $\mathfrak{L}_1 = \mathfrak{L} \cap L^{\perp}$ is a non-zero closed $\mathcal{B}^*$-invariant subspace in $L^{\perp}$. Since $L^{\perp}$ is isomorphic to $(X/L)^*$, it is reflexive, so $\mathfrak{L}_1$ is closed in the $\sigma(X^*, X)$-topology. By the bipolar theorem, there is a closed $\mathcal{B}$-invariant subspace M in X such that $\mathfrak{L}_1 = M^{\perp}$. Since L is the only non-trivial $\mathcal{B}$-invariant subspace, $\mathfrak{L}_1 = L^{\perp}$. Thus $L^{\perp} \otimes L = \mathfrak{C}_L \cap \mathcal{F}(X) \subseteq \mathcal{B}$.

Under the isomorphism of $\mathfrak{C}_L$ and $B(X/L, L)$, $\mathfrak{C}_L \cap \mathcal{F}(X)$ and $\mathfrak{C}_L \cap \mathcal{K}(X)$ correspond to $\mathcal{F}(X/L, L)$ and $\mathcal{K}(X/L, L)$, respectively. It follows from Grothendieck's theorem that the approximation property of L implies the density of $\mathcal{F}(Y, L)$ in $\mathcal{K}(Y, L)$, for any Banach space Y. Therefore, since $\mathcal{B}$ is norm-closed, $\mathfrak{C}_L \cap \mathcal{K}(X) \subseteq \mathcal{B}$. ∎

For the case where $X = H$ is a Hilbert space, Corollary 4.8(ii) allows us to obtain a description of norm-closed operator algebras of compact operators with only one non-trivial invariant subspace. We shall use the symbol $L^{\perp}$ for the orthogonal complement of L in H.

THEOREM 4.9: *If a norm-closed algebra $\mathcal{B}$ of compact operators on a Hilbert space H has only one non-trivial invariant subspace L, then*

$$B = \mathfrak{D} + (\mathfrak{C}_L \cap \mathcal{K}(H)),$$

where the algebra $\mathfrak{D}$ *consists of compact operators of the form* $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ *with respect to the decomposition* $H = L \oplus L^\perp$ *and*
either

(i) $\mathfrak{D}$ *is isomorphic to* $\mathcal{K}(L) \oplus \mathcal{K}(L^\perp)$*;*

or

(ii) *there exists a closed, densely defined, injective operator* T *from* $L^\perp$ *into* L *such that* $\mathrm{Im}(T)$ *is dense in* L*,*

$$A_2 D(T) \subseteq D(T) \quad \textit{and} \quad A_1 T = T A_2 \quad \textit{for } A \in \mathfrak{D}.$$

Proof: Clearly, in the block-matrix form $\mathfrak{C}_L$ coincides with the set of all upper triangular matrices $\begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}$. By Corollary 4.8(ii), $\mathcal{B}$ contains the set $\mathfrak{N} = \mathfrak{C}_L \cap \mathcal{K}(H)$ of all compact operators in $\mathfrak{C}_L$. Hence $\mathcal{B} = \mathfrak{D} + \mathfrak{N}$, where $\mathfrak{D}$ is a norm closed algebra which consists of block-diagonal operators.

Let Q be the projection on $L^\perp$ and consider the representations π: $A \to A|L$ and ρ: $A \to QA|L^\perp$ of $\mathcal{B}$ on L and $L^\perp$. Then $\pi(\mathcal{B}) = \pi(\mathfrak{D}) \subseteq \mathcal{K}(L)$, $\rho(\mathcal{B}) = \rho(\mathfrak{D}) \subseteq \mathcal{K}(L^\perp)$.

Suppose that $J_\rho = \mathrm{Ker}(\rho|\mathfrak{D}) \neq \{0\}$. Since $\pi(\mathfrak{D})$ is transitive on L, $\pi(J_\rho)$ is a transitive, norm-closed subalgebra of $\mathcal{K}(L)$. Hence $\pi(J_\rho) = \mathcal{K}(L)$ and it follows that $\mathfrak{D}$ is isomorphic to $\mathcal{K}(L) \oplus \mathcal{K}(L^\perp)$. The same is true if $J_\pi = \mathrm{Ker}(\pi|\mathfrak{D}) \neq \{0\}$.

Suppose now that $J_\pi = J_\rho = 0$. Since $\mathfrak{D}$ is a closed algebra of compact operators, $\pi|\mathfrak{D}$ and $\rho|\mathfrak{D}$ are $\mathcal{F}$-representations of $\mathfrak{D}$ and part (ii) follows from Lemma 3.1(ii). ∎

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